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# A BOUNDARY ELEMENT FORMULATION FOR PLANAR TIME-DEPENDENT INELASTIC DEFORMATION OF PLATES WITH CUTOUTS

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**Abstract**—A boundary element formulation for planar, time-dependent, inelastic deformation problems for bodies with cutouts is presented in this paper. A stress function description for these nonlinear problems leads to a non-homogeneous biharmonic equation for the stress function rate. An integral representation of the solution uses modified kernels which guarantee that the cutout boundary is traction free for all time. This incorporation of the effect of the cutout on the stress field into the kernels leads to an accurate determination of stresses in the near field of the cutout. Illustrative analytical examples for circular plates with circular cutouts are presented in this paper. In a companion paper[14], numerical solutions are presented for problems of finite plates with very narrow elliptic cutouts. These problems are of considerable importance in inelastic fracture.

## INTRODUCTION

The boundary element method (BEM—also called the boundary-integral equation method) has been applied quite extensively to problems of elasticity and elastic fracture mechanics (see, e.g. Refs. [1, 2]), but applications to nonlinear inelasticity problems have been relatively few. The authors of this paper together with others, have been interested in the application of the BEM to problems of time-dependent inelastic deformation[3–7]. Planar problems are considered in Refs. [3–6] and plate bending problems in Ref. [7]. In these papers, the governing differential equations are written in terms of rates and material behavior is assumed to be described by one of a new class of combined creep-plasticity constitutive models using state variables, proposed recently by several researchers. The constitutive model due to Hart[8, 9] has been used in most of the numerical examples presented in these papers. (See Refs. [3, 4] for references to other such constitutive models.)

This approach appears very useful since these new constitutive models attempt to describe inelastic deformation in metals more faithfully than is possible with traditional models which separate plastic and creep strains, while, in most cases, their mathematical structure permits a particularly simple boundary element scheme. Thus, this approach seems to combine the twin advantages of using a more realistic constitutive model to describe material behavior, together with an efficient scheme for the solution of boundary value problems of practical importance.

The mathematical structure of many of the state variable models of inelastic deformation can be summarized by the following equations

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^n_{ij} + \dot{\epsilon}^T_{ij}, \quad \dot{\epsilon}^n_{ij} = h_{ij}(\sigma_{ij}, q^{(k)}_{ij}, T) \tag{1,2}$$

$$\dot{q}^{(k)}_{ij} = g_{ij}(\sigma_{ij}, q^{(k)}_{ij}, T), \quad \dot{\epsilon}^n_{kk} = 0. \tag{3,4}$$

Here $\dot{\epsilon}^e_{ij}$, $\dot{\epsilon}^n_{ij}$ and $\dot{\epsilon}^T_{ij}$ are the elastic, non-elastic and thermal strain rates respectively, $\sigma_{ij}$ is the stress tensor, $T$ is the temperature and $q^{(k)}_{ij}$ are state variables. The number of state variables varies in the different models and they can be scalars or tensors. These state variables are assumed to completely characterize the present deformation state of the material and the history dependence of the rate of non-elastic strain up to the current time is assumed to be completely taken into account by their current values. It is important to note that the rates of the non-elastic strain and state variables at any time depend only on the current values of the stress, state variables and temperature. The usual equations of time-hardening and strain-hardening creep also fit into this general format.

The kernels used in the integral equations in Refs. [3–6] are the usual Kelvin traction and displacement functions for unit point loads in an *infinite* region. In the numerical procedure, boundary conditions along outside as well as inside boundaries (in multiply connected bodies) are satisfied at discrete points. This formulation generally gives very good numerical results except in a narrow region along the boundary. In non-elastic problems, the nonelastic strain rates are typically proportional to high powers of stress and the nonelastic strain rates over the entire region contribute to the rates of stress and displacement. Moreover, in problems with regions of high concentration of stresses and stress gradients, such regions provide nearly all the nonelastic contribution to the stress and displacement rates. It is imperative, therefore, that stresses in these regions be calculated sufficiently accurately if the time histories of stresses and displacements are to be obtained with acceptable accuracy.

Inelastic deformation problems for plates with sharp cutouts are of considerable practical interest. Some regions near these cutouts are typically regions of high stress concentration. A numerical procedure using the boundary element method with Kelvin kernels (e.g. [5]) would usually require a large number of boundary elements in such regions in order to obtain the stresses accurately near the cutouts. This number can become prohibitive for problems of plates with cracks and this may lead to numerical difficulties.

An alternative BEM formulation for planar *elastic* problems for bodies with cutouts has been presented recently[10–12]. In this approach, the Kelvin kernels are augmented so that the new kernels are the fundamental solutions of the governing differential equations for infinite regions with cutouts. Thus, the effect of the cutout on the stress and displacement fields is incorporated into the kernels of the integral equations. Use of an appropriate kernel can guarantee, for example, a traction free crack in a given region and discrete modelling of the crack boundary is no longer necessary. The methods of Muskhelishvili[13] are used to obtain these augmented kernels and this approach leads to an accurate determination of stresses, especially in the near field of the cutout.

The linear superposition principle in valid in linear elasticity and is used to advantage in this alternative formulation for elastic problems. Also, the augmentation of Kelvin kernels for Navier's displacement equations is a natural in elastic problems. In Refs. [11, 12], for example, an appropriate layer of body force is applied on the outside boundary of a body in order to satisfy the boundary conditions. Physical body forces (e.g. centrifugal forces) if present, can also be taken care of in a similar fashion. The subject of the present paper, however, is nonlinear inelasticity problems where the presence of inelastic strains causes the elastic strain fields to become incompatible. The total strains, of course, must be compatible.

A BEM formulation using augmented kernels, suitable for the solution of planar time-dependent inelastic problems, is presented in this paper. A stress function description is used and writing the equations in terms of rates leads to a nonhomogeneous biharmonic equation for the stress function rate. The nonhomogeneous term in this equation results from the presence of nonelastic strains. This equation is transformed into an integral equation by using, as kernels, two fundamental solutions of the biharmonic equation. The unknown functions are two concentration layers on the boundary of the body and these are obtained from the traction boundary conditions of the problem. The augmented kernels that guarantee traction free cutouts are obtained by Muskhelishvili's methods. The explicit augmented kernels for circular cutouts are derived and these are used in an analytical illustrative example for inelastic deformation of a circular disc with a circular cutout. In a companion paper[14], the kernels for an elliptical cutout are derived and numerical results are presented for several cracked plates in plane stress subjected to normal or shearing stresses on the boundary. The time-dependent redistribution of stress fields near the cracks are studied in these numerical examples. Either power law creep or the constitutive model due to Hart[8, 9] are used in the numerical calculations. Other constitutive models having the mathematical structure of eqns (1)–(4) can be easily incorporated into the computer program that generates these numerical results.

## GOVERNING DIFFERENTIAL EQUATIONS

A planar body is considered with the $x_1$ and $x_2$ axes in the plane of the body and the $x_3$ axis normal to it. A stress function $\Phi$ is defined in the usual way

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1 \partial x_2} \tag{5}$$

where $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12}$ are the stress components.

The strain rates are decomposed into elastic and nonelastic components as in eqn (1) (the thermal strain rates are set to zero for simplicity). Using Hooke's law to relate the rates of elastic strains and stresses and the compatibility equation in rate form

$$\frac{\partial^2 \dot{\epsilon}_{11}^e}{\partial x_2^2} + \frac{\partial^2 \dot{\epsilon}_{22}^e}{\partial x_1^2} - 2\frac{\partial^2 \dot{\epsilon}_{12}^e}{\partial x_1 \partial x_2} = -\left[\frac{\partial^2 \dot{\epsilon}_{11}^n}{\partial x_2^2} + \frac{\partial^2 \dot{\epsilon}_{22}^n}{\partial x_1^2} - 2\frac{\partial^2 \dot{\epsilon}_{12}^n}{\partial x_1 \partial x_2}\right] \tag{6}$$

results in an inhomogeneous biharmonic equation for the rate of the stress function

$$\nabla^4 \dot{\Phi} = C^{(n)}. \tag{7}$$

The function $C^{(n)}$ has the forms

$$C^{(n)} = -E\left[\frac{\partial^2 \dot{\epsilon}_{11}^n}{\partial x_2^2} + \frac{\partial^2 \dot{\epsilon}_{22}^n}{\partial x_1^2} - 2\frac{\partial^2 \dot{\epsilon}_{12}^n}{\partial x_1 \partial x_2}\right] \text{ for plane stress } (\sigma_{33} = 0)$$

$$C^{(n)} = -\frac{E}{1-\nu^2}\left[\frac{\partial^2 \dot{\epsilon}_{11}^n}{\partial x_2^2} + \frac{\partial^2 \dot{\epsilon}_{22}^n}{\partial x_1^2} - \frac{2\partial^2 \dot{\epsilon}_{12}^n}{\partial x_1 \partial x_2} + \nu \nabla^2(\dot{\epsilon}_{11}^n + \dot{\epsilon}_{22}^n)\right] \text{ for plane strain } (\epsilon_{33} = 0)$$

with $E$ and $\nu$ the Young's modulus and Poisson's ratio, respectively, of the material of the body and $\nabla$ the gradient operator.

The admissible boundary conditions for the problems considered in this paper are prescribed histories of traction on the outside boundary of the body.

## BOUNDARY ELEMENT FORMULATION

### *Simply connected body*

The biharmonic eqn (7) can be transformed into an integral equation by using two singular solutions of this equation, $s^2 \ln s$ and its normal derivative at a field point, $(\partial/\partial n_Q)(s^2 \ln s)$

$$8\pi\dot{\Phi}(p) = \oint_{\partial B} (s^2 \ln s)_{pQ} C_1(Q)\, \mathrm{d}c_Q + \oint_{\partial B} \frac{\partial}{\partial n_Q}(s^2 \ln s)_{pQ} C_2(Q)\, \mathrm{d}c_Q + \int_B (s^2 \ln s)_{pq} C^{(n)}(q)\, \mathrm{d}A_q. \tag{8}$$

Here $C_1$ and $C_2$ are unknown concentration functions to be determined from boundary conditions and $s$ is the distance between the source point $p$ (or $P$) and the field point $q$ (or Q), where lower case letters denote points inside the body $B$ and capital letters denote points on its boundary $\partial B$ (see Fig. 1).

It is convenient to rewrite this equation in terms of complex variables with a view towards conformal mapping techniques that will be used later to derive the necessary augmented kernels. This has the form (see Fig. 1).

$$8\pi\dot{\Phi}(p) = \oint_{\partial B} K_1(p, Q) C_1(Q)\, \mathrm{d}c_Q + \oint_{\partial B} K_2(p, q) C_2(Q)\, \mathrm{d}c_Q + \int_B K_1(p, q) C^{(n)}(q)\, \mathrm{d}A_q \tag{9}$$

where

$$K_1 = \mathrm{Re}\,[\bar{z}\hat{\phi}_1(z, z_0) + \hat{X}_1(z, z_0)]$$

$$K_2 = \mathrm{Re}\,[\bar{z}\hat{\phi}_2(z, z_0) + \hat{X}_2(z, z_0)]$$

$$\hat{\phi}_1(z, z_0) = (z - z_0) \ln (z - z_0)$$

$$\hat{X}_1(z, z_0) = -\bar{z}_0(z - z_0) \ln (z - z_0)$$

$$\hat{\phi}_2(z, z_0) = \frac{\partial}{\partial n_0} \hat{\phi}_1(z, z_0), \quad \hat{X}_2(z, z_0) = \frac{\partial}{\partial n_0} \hat{X}_1(z, z_0)$$

where $n_0$ is the outward normal at the field point $Z_0$ on the boundary $\partial B$, Re denotes the real part of the complex function within brackets and a superscribed bar denotes the complex conjugate.

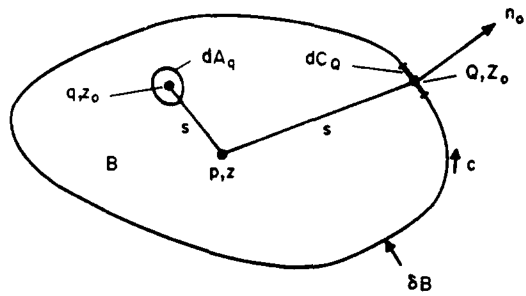


Fig. 1.

The primary interest in the problems is the determination of stresses rather than the stress function. Thus, it is convenient to write the corresponding equations for stress rates. Also, when the traction equations are written for a point $Z$ on the boundary, care must be taken to include residues, if any, that are generated from singular kernels. These matters will be given careful attention in the next section when the equations for multiply connected regions using augmented kernels are presented.

*Body with cutout*

*Augmented kernels.* The singular kernels $K_1$ and $K_2$ in eqn (9) are augmented with regular kernels so that the sum of these guarantee a traction free inner boundary in a body with cutout. The new kernels are derived by using the methods of Muskhelishvili[13]. The approach is similar to that used in Refs. [11, 12] for the analogous elastic problem and will be briefly outlined here.

Consider an infinite plane with a cutout of contour $\partial B_1$ (Fig. 2). The traction resultants $F_1$ and $F_2$ on a portion of arc $AB$ on $\partial B_1$ due to the functions $\hat{\phi}$ and $\hat{X}$ are [13, 15] ($\hat{\phi}$ can be either $\hat{\phi}_1$ or $\hat{\phi}_2$ of eqn (9) and similarly for $\hat{X}$)

$$F_1 + iF_2 = \int_A^B (\tau_1 + i\tau_2)\,dc$$
$$= -i[\hat{\phi}(Z, z_0) + \overline{Z\hat{\phi}'(Z, z_0)} + \overline{\hat{\psi}(Z, z_0)}]_A^B \quad (10)$$

where $\tau_1$ and $\tau_2$ are the components of traction at a point $Z$ in $AB$ on $\partial B_1$, $dc$ is an element of the curve $\partial B_1$ and $\psi(z, z_0) = X'(z, z_0)$, the prime denoting differentiation with respect to the variable argument $z$. If a mapping function

$$z = \omega(\xi) \quad (11)$$

can be found which maps the region on and outside $\partial B_1$ in the $z$ plane to a region on and inside an unit circle $\gamma$ in the $\xi$ plane, the expression within brackets on the right hand side of eqn (10)

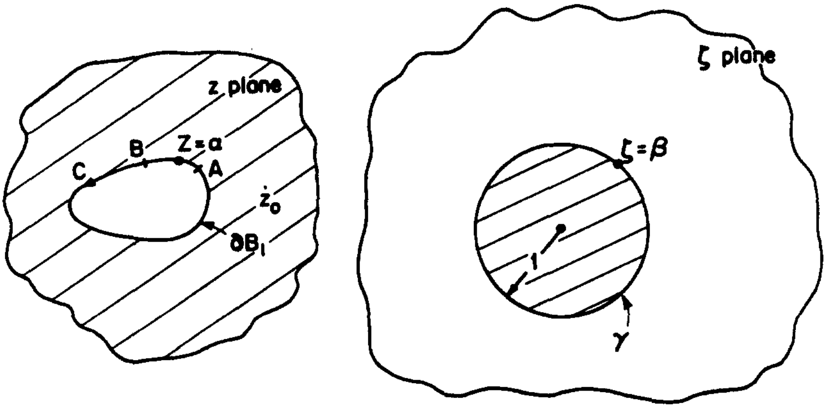


Fig. 2.

at a point $Z=\alpha$ on $\partial B_1$ can be written as

$$-F(\beta, z_0) \equiv \hat{f}(\beta, z_0) + \frac{\omega(\beta)}{\overline{\omega'(\beta)}} \overline{\hat{f}'(\beta, z_0)} + \overline{\hat{g}(\beta, z_0)} \tag{12}$$

where

$$\hat{\phi}(z, z_0) = \hat{f}(\xi, z_0)$$
$$\hat{\psi}(z, z_0) = \hat{g}(\xi, z_0)$$

the point $z=\alpha$ maps to a point $\xi=\beta$ on the unit circle $\gamma$ and $\hat{f}' = (\mathrm{d}/\mathrm{d}\xi)\hat{f}(\xi, z_0)$.

In the physical problem under consideration, the contour $\partial B_1$ must be traction free. Thus, new functions $\phi^*(z, z_0)$ and $\psi^*(z, z_0)$ must be obtained such that, the tractions due to

$$\phi(z, z_0) = \hat{\phi}(z, z_0) + \phi^*(z, z_0)$$

and

$$\psi(z, z_0) = \hat{\psi}(z, z_0) + \psi^*(z, z_0)$$

vanish on the contour $\partial B_1$. The problem, therefore, reduces to the determination of $\phi^*$ and $\psi^*$ such that

$$f^*(\beta, z_0) + \frac{\omega(\beta)}{\overline{\omega'(\beta)}} \overline{f^{*\prime}(\beta, z_0)} + \overline{g^*(\beta, z_0)} = F(\beta, z_0) \tag{13}$$

where, as before, $\phi^*(z, z_0) = f^*(\xi, z_0)$ and similarly for $g^*$.

This problem has the solution[13]

$$f^*(\xi, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{F(\beta, z_0)}{\beta - \xi} \mathrm{d}\beta - \frac{1}{2\pi i} \oint_\gamma \frac{\omega(\beta)}{\overline{\omega'(\beta)}} \frac{\overline{f^{*\prime}(\beta, z_0)}}{\beta - \xi} \mathrm{d}\beta \tag{14}$$

$$g^*(\xi, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{\overline{F(\beta, z_0)}\,\mathrm{d}\beta}{\beta - \xi} - \frac{1}{2\pi i} \oint_\gamma \frac{\overline{\omega(\beta)}}{\omega'(\beta)} \frac{f^{*\prime}(\beta, z_0)}{\beta - \xi} \mathrm{d}\beta. \tag{15}$$

The specific example for a circular cutout follows. If the contour $\partial B_1$ is a unit circle in the $z$ plane, the appropriate mapping function is

$$\omega(\xi) = 1/\xi$$

In this case, for the functions $\hat{\phi}_1$ and $\hat{X}_1$ of eqn (9),

$$F(\beta, z_0) = -\left(\frac{1}{\beta} - z_0\right) \ln\left(\frac{1}{\beta} - z_0\right) - \frac{1}{\beta} \ln(\beta - \bar{z}_0) - \frac{1}{\beta} + z_0 \ln(\beta - \bar{z}_0) + z_0. \tag{16}$$

The second term in eqn (14) vanishes and the second term in eqn (15) becomes $\xi^3 f^{*\prime}(\xi, z_0)$. Solving for $f^*$ and $g^*$, the kernels, within additive functions of $z_0$, are

$$\phi_1(z, z_0) = (z - z_0) \ln(z - z_0) + z_0 \ln\left(\frac{1}{z} - \bar{z}_0\right) - z \ln\left(1 - \frac{1}{z\bar{z}_0}\right) \tag{17}$$

$$\psi_1(z, z_0) = -\bar{z}_0 \ln(z - z_0) - \left(\frac{1}{z} - \bar{z}_0\right) \ln\left(\frac{1}{z} - \bar{z}_0\right) - \frac{1}{z}(1 + \ln(-z_0))$$
$$+ \frac{1}{z} \ln\left(1 - \frac{1}{z\bar{z}_0}\right) + \frac{z_0}{z^2(1 - z\bar{z}_0)} + \frac{1}{z(z\bar{z}_0 - 1)} \tag{18}$$

and

$$\phi_2 = \frac{\partial \phi_1}{\partial n_0}, \qquad \psi_2 = \frac{\partial \psi_1}{\partial n_0}.$$

Since stresses involve derivatives of these functions with respect to $z$, the additive functions of $z_0$ are of no consequence. It can be easily verified that

$$\phi_i(\alpha, z_0) + \alpha\overline{\phi_i'(\alpha, z_0)} + \overline{\psi_i(\alpha, z_0)} = 0 \quad (i = 1, 2)$$

on any point $Z = \alpha$ on the circle $|Z| = 1$, i.e. the tractions due to these functions vanish on $\partial B_1$.

*Single valued displacement on inner boundary.* The displacement vector **u** must be single valued on the boundary of the cutout $\partial B_1$ in Fig. 3, i.e. it is required that

$$\oint_{\partial B_1} \mathrm{d}\mathbf{u} = 0. \tag{19}$$

In the presence of nonelastic strain rates in the body but with zero tractions on $\partial B_1$, this condition, in terms of the rate of the stress function, gives rise to three equations for plane strain problems.

$$\oint_{\partial B_1} \frac{\mathrm{d}}{\mathrm{d}n}(\nabla^2\dot{\Phi})\,\mathrm{d}c = \oint_{\partial B_1} \mathbf{D}^{(n)}\cdot\mathbf{n}\,\mathrm{d}c \tag{20}$$

$$\oint_{\partial B_1}\left(x_2\frac{\mathrm{d}}{\mathrm{d}n} - x_1\frac{\mathrm{d}}{\mathrm{d}c}\right)\nabla^2\dot{\Phi}\,\mathrm{d}c = \oint_{\partial B_1} x_2(\mathbf{D}^{(n)}\cdot\mathbf{n})\,\mathrm{d}c - \frac{E}{1-\nu^2}\left[\oint_{\partial B_1} \dot{\epsilon}_{11}^n\,\mathrm{d}x_1 + \dot{\epsilon}_{12}^n\,\mathrm{d}x_2 + \nu(\dot{\epsilon}_{11}^n + \dot{\epsilon}_{22}^n)\,\mathrm{d}x_1\right] \tag{21}$$

$$\oint_{\partial B_1}\left(x_1\frac{\mathrm{d}}{\mathrm{d}n} + x_2\frac{\mathrm{d}}{\mathrm{d}c}\right)\nabla^2\dot{\Phi}\,\mathrm{d}c = \oint_{\partial B_1} x_1(\mathbf{D}^{(n)}\cdot\mathbf{n})\,\mathrm{d}c + \frac{E}{1-\nu^2}\left[\oint_{\partial B_1} \dot{\epsilon}_{12}^n\,\mathrm{d}x_1 + \dot{\epsilon}_{22}^n\,\mathrm{d}x_2 + \nu(\dot{\epsilon}_{11}^n + \dot{\epsilon}_{22}^n)\,\mathrm{d}x_2\right] \tag{22}$$

where

$$D_1^{(n)} = \frac{E}{1-\nu^2}\left[-\frac{\partial\dot{\epsilon}_{22}^n}{\partial x_1} + \frac{\partial\dot{\epsilon}_{12}^n}{\partial x_2} - \nu\frac{\partial}{\partial x_1}(\dot{\epsilon}_{11}^n + \dot{\epsilon}_{22}^n)\right]$$

$$D_2^{(n)} = \frac{E}{1-\nu^2}\left[\frac{\partial\dot{\epsilon}_{12}^n}{\partial x_1} - \frac{\partial\dot{\epsilon}_{11}^n}{\partial x_2} - \nu\frac{\partial}{\partial x_2}(\dot{\epsilon}_{11}^n + \dot{\epsilon}_{22}^n)\right].$$

Note that $\nabla\cdot\mathbf{D}^{(n)} = C^{(n)}$.

The equations for plane stress have exactly the same form with $\nu$ set equal to zero.

These equations are derived in a manner analogous to the elastic case. The first of these equations is a statement of zero net rotation around the boundary $\partial B_1$, while the second and third guarantee, respectively,

$$\oint_{\partial B_1} \mathrm{d}(u_1 + x_2\omega_{12}) = 0 \text{ and } \oint_{\partial B_1} \mathrm{d}(u_2 - x_1\omega_{12}) = 0$$

where $\omega_{12} = u_{2,1} - u_{1,2}$ is the rotation in the plane of the body. For a discussion of the elastic situation, see, for example, Timoshenko and Goodier[15].

It is noted that if Fig. 3, in fact, represented a simply connected body, the field eqn (7) would be valid everywhere including the region $B_1$ and the eqns (20)–(22) for single valued displace-

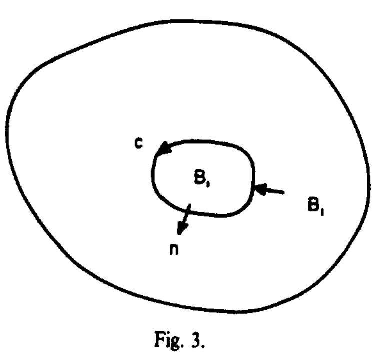


Fig. 3.

ments would be satisfied on the boundary $\partial B_1$. In fact, in such a case, eqns (20)–(22) on $\partial B_1$ can be derived directly from eqn (7) in $B_1$ by using Green's theorem and the divergence theorem, provided these theorems are applicable. This matter will be alluded to in the next section.

*Integral equations for stresses and tractions*

The stress rates in a body with a cutout (Fig. 4) are written as (with $i, j = 1, 2$)

$$8\pi\dot{\sigma}_{ij}(p) = \oint_{\partial B_2} H_{ij}^{(1)}(p, Q)C_1(Q)\,\mathrm{d}c_Q + \oint_{\partial B_2} H_{ij}^{(2)}(p, Q)C_2(Q)\,\mathrm{d}c_Q + \int_B H_{ij}^{(1)}(p, q)C^{(n)}(q)\,\mathrm{d}A_q - \oint_{\partial B_1} H_{ij}^{(1)}(p, Q)D_k^{(n)}(Q)n_k(Q)\,\mathrm{d}c_Q \quad (23)$$

where the augmented kernels $H_{ij}^{(k)}$, $k = 1, 2$, are

$$H_{11}^{(k)}(z, z_0) = \mathrm{Re}[2\phi_k'(z, z_0) - \bar{z}\phi_k''(z, z_0) - \psi_k'(z, z_0)]$$
$$H_{22}^{(k)}(z, z_0) = \mathrm{Re}[2\phi_k'(z, z_0) + \bar{z}\phi_k''(z, z_0) + \psi_k'(z, z_0)]$$
$$H_{12}^{(k)}(z, z_0) = \mathrm{Im}[\bar{z}\phi_k''(z, z_0) + \psi_k'(z, z_0)].$$

The first three terms on the right hand side are analogous to those in eqn (9). The last term represents a layer of concentration $\mathbf{n}\cdot\mathbf{D}^{(n)}$ on the cutout boundary $\partial B_1$ and is included with a

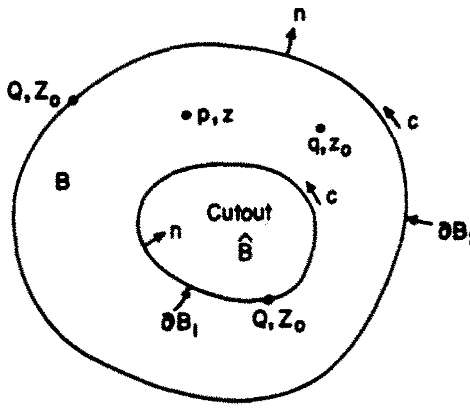


Fig. 4.

view towards obtaining single valued displacements on this boundary. This term is motivated as follows. As stated in the last section, a simply connected region would require a concentration distribution $C^{(n)}$ throughout the body $B + \hat{B}$. Thus, if the divergence theorem is applicable in the region $\hat{B}$,

$$\int_{\hat{B}} C^{(n)}\,\mathrm{d}A = \int_{\hat{B}} \nabla\cdot\mathbf{D}^{(n)}\,\mathrm{d}A = -\oint_{\partial B_1} \mathbf{n}\cdot\mathbf{D}^{(n)}\,\mathrm{d}c$$

the negative sign being a consequence of the direction of the normal $B_1$ in Fig. 4. In the body with a cutout, however, $\hat{B}$ is a forbidden zone and $\mathbf{n}\cdot\mathbf{D}^n$ is distributed on $\partial B_1$ instead. It is postulated that inclusion of this term in eqn (23) leads to satisfaction of eqn (19). While a direct proof of this conjecture has not yet been possible, correct expressions for stress rates are obtained in an analytical example of a circular disc with a circular cutout, presented later in this paper, and numerical results for a square plate with an elliptical cutout, presented in the companion paper[14], agree well with those obtained from a direct formulation of the problem with Kelvin kernels of Navier's equations[5]. Note that the corresponding elastic problem has $C^{(n)} = 0$, the last two terms of eqn (23) vanish, and the first two give the correct solution.

Using $C^{(n)}(q) = D_{k,k}^{(n)}(q)$ in the area integral in eqn (23) and applying the divergence theorem, eqn (23) can be written in a more convenient form where $\mathbf{D}^{(n)}$, with first derivatives of the strain rates, rather than $C^{(n)}$, with second derivatives, appear

$$8\pi\dot{\sigma}_{ij}(p) = \oint_{\partial B_2} H_{ij}^{(1)}(p, Q)C_1(Q)\,\mathrm{d}c_Q + \oint_{\partial B_2} H_{ij}^{(2)}(p, Q)C_2(Q)\,\mathrm{d}c_Q + \oint_{\partial B_2} H_{ij}^{(1)}(p, Q)D_k^{(n)}(Q)n_k(Q)\,\mathrm{d}c_Q - \int_B H_{ij,k_0}^{(1)}(p, q)D_{k_0}^{(n)}(q)\,\mathrm{d}A_q. \quad (24)$$

Here the $k_0$ in the last term denotes differentiation of $H_{ij}^{(1)}$ with respect to the field point.

The boundary conditions of the problem must be specified in terms of traction histories on $\partial B_2$. The traction rates $\dot{\tau}_i$ are obtained from eqn (24) by taking the limit as $p$ in $B$ approaches a point $P$ on $\partial B_2$. If $\partial B_2$ is locally smooth at $P$,

$$8\pi\dot{\tau}_i(P) = \oint_{\partial B_2} H_{ij}^{(1)}(P, Q)n_j(P)C_1(Q)\,\mathrm{d}c_Q + \oint_{\partial B_2} H_{ij}^{(2)}(P, Q)n_j(P)C_2(Q)\,\mathrm{d}c_Q + \oint_{\partial B_2} H_{ij}^{(1)}(P, Q)\,D_k^{(n)}(Q)n_j(P)n_k(Q)\,\mathrm{d}c_Q - \int_B H_{ij,k_0}^{(1)}(P, q)D_{k_0}^{(n)}(q)n_j(P)\,\mathrm{d}A_q \quad (i, j, k = 1, 2) \quad (25)$$

The first three integrals in the above equations must be interpreted in the sense of Cauchy principal values. It can be shown that the limiting process does not yield residues in the above equation for traction rates. In case of boundary stress rates, however, while the equations for normal and shearing stress rates do not yield a residue, the one for the tangential stress rate yields a residue of $4\pi C_2$ as $p$ approaches $P$ on the boundary where it is locally smooth, i.e. if $8\pi\dot{\sigma}_{cc}(p^*) = h(p^*)$ then

$$8\pi\dot{\sigma}_{cc}(P^*) = h(P^*) + 4\pi C_2(P^*)$$

where $p^*$ is infinitesimally close to $P^*$.

## ILLUSTRATIVE EXAMPLES

*Solid circular disc under uniform axisymmetric external load-elastic solution (plane strain or stress)*

The problem under consideration here is that of a circular disc of radius $b$ under an axisymmetric external load $p_0$ per unit area (Fig. 5). Since the region is simply connected, the formulation presented in eqn (9) is used with $C^{(n)} = 0$ for the elastic case. Using polar coordinates, the stresses at an inside point $p$ are

$$8\pi\sigma_{ij}(r, \alpha) = C_1(b)\int_0^{2\pi} H_{ij}^{(1)}(r, \alpha; b, \theta)b\,\mathrm{d}\theta + C_2(b)\int_0^{2\pi} H_{ij}^{(2)}(r, \alpha; b, \theta)b\,\mathrm{d}\theta \quad (26)$$

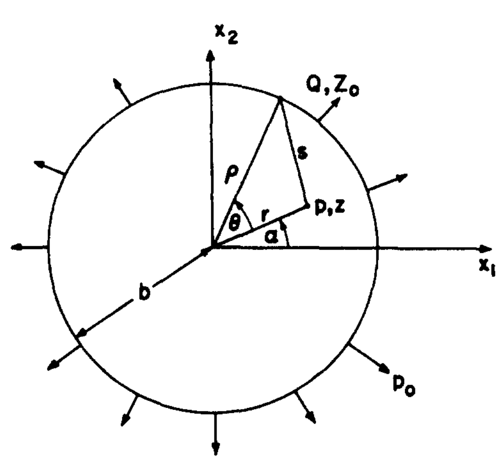


Fig. 5.

where the kernels corresponding to $K_1$ and $K_2$ of eqn (9) are

$$H^{(1)}_{22} = 2(1+\ln s) \mp \frac{r^2\cos 2\alpha + \rho^2 \cos 2(\theta+\alpha) - 2\rho r \cos(\theta+2\alpha)}{s^2}$$

$$H^{(1)}_{12} = \frac{-r^2 \sin 2\alpha - \rho^2 \sin 2(\theta+\alpha) + 2\rho r \sin(\theta+2\alpha)}{s^2}$$

$$H^{(1)}_{22} = \frac{2(\rho - r\cos\theta)}{s^2} \pm \frac{-2\rho\cos 2(\theta+\alpha) + 2r\cos(\theta+2\alpha)}{s^2}$$

$$\pm\frac{2}{s^4}\{\rho r^2 \cos 2\alpha + \rho^3 \cos 2(\theta+\alpha) - 2\rho^2 r\cos(\theta+2\alpha) - r^3 \cos 2\alpha \cos\theta$$

$$-\rho^2 r \cos 2(\theta+\alpha)\cos\theta + 2\rho r^2 \cos(\theta+2\alpha)\cos\theta\}$$

$$H^{(2)}_{12} = \frac{-2\rho \sin 2(\theta+\alpha) + 2r\sin(\theta+2\alpha)}{s^2}$$

$$+\frac{2}{s^4}\{\rho r^2 \sin 2\alpha + \rho^3 \sin 2(\theta+\alpha) - 2\rho^2 r \sin(\theta+2\alpha) - r^3 \sin 2\alpha\cos\theta$$

$$-\rho^2 r \sin 2(\theta+\alpha)\cos\theta + 2\rho r^2 \sin(\theta+2\alpha)\cos\theta\}.$$

The symbols are shown in Fig. 5 and $s^2 = \rho^2 + r^2 - 2\rho r\cos\theta$.
For a point $P$ on the boundary,

$$8\pi p_0 \cos\alpha = 8\pi\tau_1(b,\alpha)$$

$$= C_1(b)\int_0^{2\pi} [H^{(1)}_{11}(b,\alpha;b,\theta)\cos\alpha + H^{(1)}_{12}(b,\alpha;b,\theta)\sin\alpha]b\,d\theta$$

$$+ C_2(b)\int_0^{2\pi} [H^{(2)}_{11}(b,\alpha;b,\theta)\cos\alpha + H^{(2)}_{12}(b,\alpha;b,\theta)\sin\alpha]b\,d\theta \tag{27}$$

$$8\pi p_0 \sin\alpha = 8\pi\tau_2(b,\alpha)$$

$$= C_1(b)\int_0^{2\pi} [H^{(1)}_{12}(b,\alpha;b,\theta)\cos\alpha + H^{(1)}_{22}(b,\alpha;b,\theta)\sin\alpha]b\,d\theta$$

$$+ C_2(b)\int_0^{2\pi} [H^{(2)}_{12}(b,\alpha;b,\theta)\cos\alpha + H^{(2)}_{22}(b,\alpha;b,\theta)\sin\alpha]b\,d\theta. \tag{28}$$

The nonvanishing entegrals of these kernels, used in the equations, are given in Table 1. Using these, both eqns (27) and (28) give

$$2p_0 = C_1 b(1+\ln b) + C_2 \tag{29}$$

and, from eqns (26)

$$\sigma_{11}(r,\alpha) = (1/2)[C_1 b(1+\ln b) + C_2] = p_0$$

$$\sigma_{22}(r,\alpha) = p_0, \quad \sigma_{12}(r,\alpha) = 0.$$

The stresses on the boundary can be obtained from eqn (26) by taking the limit $p \to P$. In this case, the appropriate residues $4\pi\sin^2\alpha C_2$, $4\pi\cos^2\alpha C_2$ and $-4\pi\sin\alpha\cos\alpha C_2$ (corresponding to $4\pi C_2$ for the tangential stress $\sigma_{\theta\theta}$) must be included. This gives, for example,

$$8\pi\sigma_{11}(b,\alpha) = C_1 4\pi b(1+\ln b)b + C_2 2\pi(1+\cos 2\alpha) + C_2 4\pi\sin^2\alpha$$

and finally

$$\sigma_{11}(b,\alpha) = \sigma_{22}(b,\alpha) = p_0, \quad \sigma_{12}(b,\alpha) = 0$$

as expected.

Table 1. Non-vanishing integrals of kernels [Ref. 16] for solid circular disc

| SIGN | | $f(\theta)$ | $\int_0^{2\pi} f(\theta)\,d\theta$ | |
|---|---|---|---|---|
| $H_{11}^{(1)}$ | $H_{22}^{(1)}$ | | $\rho > r$ | $\rho = r$ |
| + | + | $2(1 + \ln s)$ | $4\pi(1 + \ln \rho)$ | $4\pi(1 + \ln r)$ |
| $H_{11}^{(2)}$ | $H_{22}^{(2)}$ | | | |
| + | + | $\frac{2(\rho - r\cos\theta)}{s^2}$ | $\frac{4\pi}{\rho}$ | $\frac{2\pi}{r}$ |
| + | − | $\frac{-2\rho\cos 2(\theta+\alpha) + 2r\cos(\theta+2\alpha)}{s^2}$ | 0 | $\frac{2\pi}{r}\cos 2\alpha$ |
| $H_{12}^{(2)}$ | | $\frac{-2\rho\sin 2(\theta+\alpha) + 2r\sin(\theta+2\alpha)}{s^2}$ | 0 | $\frac{2\pi}{r}\sin 2\alpha$ |

*Circular disc with concentric circular cutout under uniform axisymmetric external load–inelastic plane stress solution*

A circular disc of radius $b$ with a concentric circular cutout of unit radius is subjected to an axisymmetric external load history $p_0(t)$ (Fig. 6). The governing equation in polar coordinates for this problem is

$$\nabla^4\dot{\Phi} = C^{(n)} = \frac{E}{r}\frac{d}{dr}\left[\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n - r\frac{d\dot{\epsilon}_{\theta\theta}^n}{dr}\right] \tag{30}$$

and

$$D_r^{(n)} = \frac{E}{r}\left(\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n - r\frac{d\dot{\epsilon}_{\theta\theta}^n}{dr}\right), \quad D_\theta^{(n)} = 0. \tag{31}$$

The equation for stress rates (23), with the source point $p$ on the $x_1$ axis (this can be done without loss of generality for this axisymmetric problem) and $C_2(b) = 0$ (using only $C_1(b)$ is sufficient here because of axisymmetry) give

$$8\pi\dot{\sigma}_{ij}(r) = C_1(b)\int_0^{2\pi} H_{ij}^{(1)}(r; b, \theta)b\,d\theta + \int_1^b\int_0^{2\pi} H_{ij}^{(1)}(r; \rho, \theta)C^{(n)}(\rho)\rho\,d\theta\,d\rho$$
$$+ D_r^{(n)}(1)\int_0^{2\pi} H_{ij}^{(1)}(r; 1, \theta)\,d\theta. \tag{32}$$

For this point on the $x_1$ axis, $\sigma_{11} = \sigma_{rr}$, $\sigma_{22} = \sigma_{\theta\theta}$ and $\sigma_{12} = \sigma_{r\theta}$. The augmented kernels $H_{ij}$ are obtained from eqn (23) using the stress functions $\phi_1$ and $\psi_1$ from eqn (18). They are rather lengthy and will not be given here.

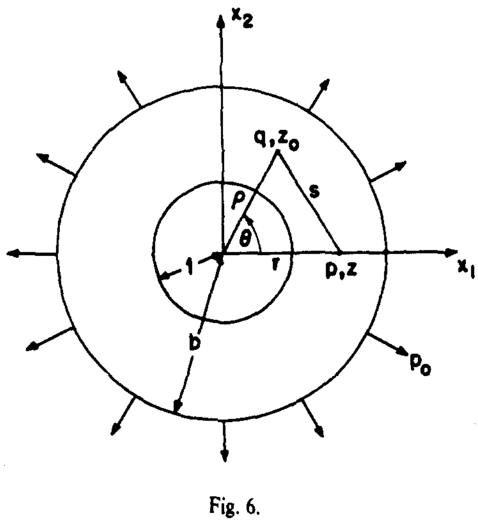


Fig. 6.

The traction equations for a boundary point $P$ are

$$8\pi\dot{p}_0 = 8\pi\dot{\tau}_1(b) = C_1(b)\int_0^{2\pi} H_{11}^{(1)}(b;b,\theta)b\,d\theta + \int_1^b\int_0^{2\pi} H_{11}^{(1)}(b;\rho,\theta)C^{(n)}(\rho)\rho\,d\theta\,d\rho$$
$$+ D_r^{(n)}(1)\int_0^{2\pi} H_{11}^{(1)}(b;1,\theta)\,d\theta \tag{33}$$

and the expression for $0 = 8\pi\dot{\tau}_2(b)$ is similar with $H_{11}^{(1)}$ replaced by $H_{12}^{(1)}$ everywhere.

The nonvanishing integrals of the kernels, in this case, are listed in Table 2. Using these, eqn (32) on the boundary can be solved for $C_1(b)$ to give

$$C_1(b) = \frac{2b\dot{p}_0}{(b^2-1)(1+\ln b)} - D_r^{(n)}(b) - \frac{E\dot{\epsilon}_{\theta\theta}^n(b)}{b(1+\ln b)} - \frac{E}{b(b^2-1)(1+\ln b)}\int_1^b \left(\frac{\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n}{\rho} + \rho\dot{\epsilon}_{zz}^n\right)d\rho \tag{32}$$

Substituting for $C_1(b)$ into eqn (32) gives the equations for stress rates

$$\dot{\sigma}_{rr}(r) = \frac{E}{2}\left\{\int_1^r \frac{\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n}{\rho}\,d\rho - \frac{(r^2-1)}{(b^2-1)}\frac{b^2}{r^2}\int_1^b \frac{(\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n)}{\rho}\,d\rho\right\}$$
$$+\frac{E}{2r^2}\left\{\int_1^r \rho\dot{\epsilon}_{zz}^n\,d\rho - \frac{(r^2-1)}{(b^2-1)}\int_1^b \rho\dot{\epsilon}_{zz}^n\,d\rho\right\} + \dot{p}_0\frac{(r^2-1)}{(b^2-1)}\frac{b^2}{r^2}$$

$$\dot{\sigma}_{\theta\theta}(r) = \frac{E}{2}\left\{\int_1^r \frac{\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n}{\rho}\,d\rho - \frac{(r^2+1)}{(b^2-1)}\frac{b^2}{r^2}\int_1^b \frac{(\dot{\epsilon}_{rr}^n - \dot{\epsilon}_{\theta\theta}^n)}{\rho}\,d\rho\right\}$$
$$-\frac{E}{2r^2}\left\{\int_1^r \rho\dot{\epsilon}_{zz}^n\,d\rho + \frac{(r^2+1)}{(b^2-1)}\int_1^b \rho\dot{\epsilon}_{zz}^n\,d\rho\right\} - E\dot{\epsilon}_{\theta\theta}^n + \dot{p}_0\frac{(r^2+1)}{(b^2-1)}\frac{b^2}{r^2}$$

$$\dot{\sigma}_{r\theta} = 0$$

which were derived earlier by direct methods[17].

Table 2. Non-vanishing integrals of kernels [Ref. 16] for annular disc

| SIGN | | | $\int_0^{2\pi} f(\theta)\,d\theta$ | | |
|---|---|---|---|---|---|
| $H_{11}^{(1)}$ | $H_{22}^{(1)}$ | | $\rho > r$ | $\rho = r$ | $\rho < r$ |
| + | + | $2(1+\ln s)$ | $4\pi(1+\ln \rho)$ | $4\pi(1+\ln r)$ | $4\pi(1+\ln r)$ |
| − | + | $\frac{r(r-\rho\cos\theta)}{s^2}$ | 0 | $\pi$ | $2\pi$ |
| − | + | $\frac{\rho^2\cos 2\theta - \rho r\cos\theta}{s^2}$ | 0 | $-\pi$ | $-2\pi\rho^2/r^2$ |
| − | + | $\frac{2(1+\ln\rho)}{r^2}$ | $\frac{4\pi(1+\ln\rho)}{r^2}$ | $\frac{4\pi(1+\ln r)}{r^2}$ | $\frac{4\pi(1+\ln\rho)}{r^2}$ |

## CONCLUSIONS

A boundary element formulation using augmented kernels is presented here for problems of planar, inelastic deformation of plates with cutouts. In this approach, two fundamental solutions of the biharmonic equation are augmented so that the resultant kernels yield the stress distributions for point concentrations in an infinite plate with a traction free cutout. Thus, the effect of the cutout on the stress field is incorporated into the kernels and the cutout boundary need not be modelled discretely in a numerical application. The specific kernels for a plate with a circular cutout are derived. Analytical illustrative examples for a uniform circular plate undergoing elastic deformation and a circular plate with a circular cutout undergoing inelastic deformation are carried out and the formulation is shown to yield the correct expressions for stresses in these cases. Numerical solutions for finite plates with elliptic cutouts, with applications to inelastic fracture mechanics, are presented in a companion paper[14].

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